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ON THE CARNOT THEOREM IN THE THEORY OF IMPULSIVE MOTION OF MECHANICAL SYSTEMS

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The kinetic energy variation of a system subjected to impact under conditions of the generalized Carnot theorem [1] is investigated. Proof of the theorem is given in the case of acatastatic links. It is shown that the second part of that theorem extends to systems whose interaction with the links is defined by properties which include the known conditions of ideality of the latter and, also, of zero virtual work of link reaction impulses during the regeneration time.

1. Let the constraints on the motion of a mechanical system be of the form of ideal links defined by independent equations of the form

$$\sum_{k=1}^{n}(a_{sk}x_k^{\cdot}+b_{sk}y_k^{\cdot}+c_{sk}z_k^{\cdot})+d_s=0 \quad (s=1,\ldots,l) \tag{1.1}$$

where $a_{sk}, b_{sk}, c_{sk}, d_s$ are continuous and continuously differentiable functions of time t and coordinates x_k, y_k, z_k $(k=1,\ldots,n)$, $(x^{\cdot}=dx\,/\,dt)$. of the system mass points.

We introduce velocity vectors $\mathbf{v}_k=(x_k^{\cdot}, y_k^{\cdot}, z_k^{\cdot})$ of points and vectors $\mathbf{N}_{sk}=(a_{sk}, b_{sk}, c_{sk})$ $(s=1,\ldots,l;\ k=1,\ldots,n)$, and write Eqs. (1.1) as follows:

$$\sum_k \mathbf{N}_{sk}\mathbf{v}_k+d_s=0 \quad (s=1,\ldots,l) \tag{1.2}$$

The condition of ideality of links implies that the virtual displacements $\delta\mathbf{r}_k$ on which the work of link reactions is zero, satisfy the equalities

$$\sum_k \mathbf{N}_{sk}\delta\mathbf{r}_k=0 \quad (s=1,\ldots,l) \tag{1.3}$$

Let us consider the kinetic energy variation of a system constrained by links that remain after impact (the first part of the Carnot theorem). The velocity variation of points of the system resulting from the application of links corresponds to the effect of reaction impact impulses $\mathbf{R}_k$

$$m_k(\mathbf{v}_{k+}-\mathbf{v}_{k-})=\mathbf{R}_k, \quad \mathbf{R}_k=\sum_{s=1}^{l}\lambda_s\mathbf{N}_{sk} \quad (k=1,\ldots,n) \tag{1.4}$$

where m_k are masses of points of the system, $\mathbf{v}_{k-}$ and $\mathbf{v}_{k+}$ are the velocities of points of the system before and after impact, and λ_s $(s=1,\ldots,l)$ are undefined coefficients.

Since the links remain after impact, velocities $\mathbf{v}_{k+}$ of points of the system satisfy Eqs. (1.2). Equations (1.2) for $\mathbf{v}_{k+}$ and (1.4) enable us to determine $(3n+l)$ unknowns $x_k^{\cdot}, y_k^{\cdot}, z_k^{\cdot}, \lambda_s$, if the state of the system before the impact is specified.

The system of linear equations for the undefined coefficients λ_s is of the form

$$\sum_{k=1}^{n} \frac{1}{m_k} \sum_{p=1}^{l} \lambda_p \mathbf{N}_{pk} \mathbf{N}_{sk} = d_s + \sum_{k=1}^{n} \mathbf{N}_{sk} \mathbf{v}_{k-} \quad (s = 1, \ldots, l) \tag{1.5}$$

which implies that the reactions are uniquely determined, if the system of Eqs. (1.5) is compatible and has a unique solution.

We represent the motion of each point as a composite one. As the carrier velocity $(\mathbf{v}_k^e)$ we take the velocities which the points would have if the system was at rest $(\mathbf{v}_{k-} = 0)$ but constrained by links (1.1), i.e. $\mathbf{v}_k^e$ defined by Eqs. (1.2). The relative velocity vectors $\mathbf{v}_{k-}^{\circ}$ and $\mathbf{v}_{k+}^{\circ}$ of points before and after impact are, then, defined by $\mathbf{v}_{k-}^{\circ} = \mathbf{v}_k - \mathbf{v}_k^e$ and $\mathbf{v}_{k+}^{\circ} = \mathbf{v}_{k+} - \mathbf{v}_k^e$, respectively. The remainder of Eqs. (1.2) for $\mathbf{v}_{k+}$ and $\mathbf{v}_k^e$ yields the equalities

$$\sum_{k=1}^{n} \mathbf{N}_{sk} \overset{\circ}{\mathbf{v}}_{k+} = 0 \quad (s = 1, \ldots, l) \tag{1.6}$$

Equations (1.4) are also valid for the relative velocities. The scalar multiplication of each of Eqs. (1.4) by $\mathbf{v}_{k+}^{\circ}$ and subsequent summation yields

$$\sum_{k=1}^{n} (\overset{\circ}{\mathbf{v}}_{k+} - \overset{\circ}{\mathbf{v}}_{k-}) \overset{\circ}{\mathbf{v}}_{k+} = \sum_{k=1}^{n} \mathbf{R}_k \overset{\circ}{\mathbf{v}}_{k+} \tag{1.7}$$

Taking into account (1.6) we conclude that the right-hand side of (1.7) is zero hence

$$\frac{1}{2} \sum_k m_k \overset{\circ}{v}{}_{k-}^2 - \frac{1}{2} \sum_k m_k \overset{\circ}{v}{}_{k+}^2 = \frac{1}{2} \sum_k m_k (\mathbf{v}_{k-} - \mathbf{v}_{k+})^2 \tag{1.8}$$

Equality (1.8) shows that the kinetic energy of relative velocities lost at the sudden imposition of the retained ideal acatastatic links is equal to the energy of lost velocities.

When $d_s \equiv 0$ (catastatic links [2]) from (1.8) we obtain the first part of the generalized Carnot theorem.

2. The second part of the generalized Carnot theorem assumes that the system motion is compatible with the pre-impact links. Hence the point velocities $\mathbf{v}_{k-}$ before the impact satisfy Eqs. (1.2) for the links. Impact reactions can be due either to the effect of active impact forces applied to points of the system or be the result of impulsive variation of links [2]. In both cases the model of impact interaction in discrete mechanical systems must be supplemented by assumptions related to the physical properties of the mechanical system and of constraints represented by the links, since the removal of the latter occurs either as the result of elastic interaction with them or of their destruction. In the first case we use the known concept of elastic impact interaction with the link as a process consisting of two phases: accumulation in the link of impact reaction impulses, and the subsequent recovery and restitution of accumulated impulses to the system. We limit the action of active impact forces to the duration of the first phase.

During the first phase the links are present and the impulses $\mathbf{R}_k$ are generated. Denoting the velocities of points by $\mathbf{u}_k$ we obtain

$$m_k(\mathbf{u}_k - \mathbf{v}_{k-}) = \mathbf{S}_k + \mathbf{R}_k \qquad (k = 1, \ldots, n) \tag{2.1}$$

where Eq. (1.2) is also valid for $\mathbf{u}_k$, and $\mathbf{S}_k$ are impusles of active shock forces.

In the course of the second phase only impact reactions whose impulses we denote by $\mathbf{R}_k^*$ act on the system. Then

$$m_k(\mathbf{v}_{k+} - \mathbf{u}_k) = \mathbf{R}_k^* \qquad (k = 1, \ldots, n) \tag{2.2}$$

The direction of impact reactions during the recovery phase is generally determined by vector $\mathbf{N}_{sk}$. Using Newton's hypothesis on the possibility of defining the elastic interaction of mass points of the system with the links by the restitution coefficients $(0 \leqslant \varkappa_{sk} \leqslant 1)$ and the disintegration by the velocity conservation coefficients $(-1 \leqslant \varkappa_{sk} < 0)$, we have

$$\mathbf{R}_k^* = \sum_{s=1}^{l} \varkappa_{sk}\lambda_s\mathbf{N}_{sk}, \quad -1 \leqslant \varkappa_{sk} \leqslant 1 \tag{2.3}$$

Hence, if the links are to remain ideal at impact, it is necessary that the virtual work of shock reaction impulses during the restitution phase is zero, i.e.

$$\sum_k \mathbf{R}_k^*\delta\mathbf{r}_k = 0 \quad \Big(\sum_k \mathbf{R}_k\delta\mathbf{r}_k = 0\Big) \tag{2.4}$$

R e m a r k. A comparison of (2.3) and (1.3) shows that condition (2.4) of ideality at impact is satisfied if the characteristics of impact interaction of all particles with each link are the same, i.e. $\varkappa_{s1} = \ldots = \varkappa_{sn} = \varkappa_s$.

From equalities (2.1) and (2.2) we have

$$m_k(\mathbf{v}_{k+} - \mathbf{v}_{k-}) = \mathbf{S}_k + \mathbf{R}_k + \mathbf{R}_k^* \quad (k = 1, \ldots, n) \tag{2.5}$$

In proving the theorem second part it is usual to take as the input equations the equation of impuslive motion [3] which are obtained from (2.5), when the active impact forces are balanced by impact reactions acting during the first phase, i.e. $\mathbf{S}_k = -\mathbf{R}_k$. Otherwise it is necessary to consider velocities $\mathbf{u}_k$ instead of $\mathbf{v}_{k-}$.

Let us assume that the links remain ideal under impact, i.e. that condition (2.4) is satisfied. The scalar multiplication of each of equalities (2.5) by vectors $\mathring{\mathbf{v}}_{k-}$ with allowance for the equality $\mathbf{S}_k = -\mathbf{R}_k$ yields for their sum

$$\frac{1}{2}\sum_k m_k \mathring{v}_{k+}^2 - \frac{1}{2}\sum_k m_k \mathring{v}_{k-}^2 - \frac{1}{2}\sum_k m_k(\dot{\mathbf{v}}_{k+} - \dot{\mathbf{v}}_{k-})^2 = \sum_k \mathbf{R}_k^*\dot{\mathbf{v}}_{k-} \tag{2.6}$$

Since the relative velocities $\mathring{\mathbf{v}}_{k-}$ are proportional to virtual displacements, the right-hand side of equality (2.6), under condition (2.4), is zero.

The second part of the Carnot theorem has been, thus generalized as follows. Under the impact of active forces followed by liberation of the system of acatastatic links ideal under impact, the acquired kinetic energy of relative velocities is equal to the kinetic energy of acquired velocities.

We also obtain the left-hand side of Eq. (2.6) when the impact effect on the system is due to impulsive links [2]. In such case functions d_s in Eqs. (1.1) have a first order discontinuity and assume values which we denote by d_s^*. The change of links causes the shock. At the end of the first phase velocity $\mathbf{u}_k^*$ of points satisfy the equations

$$m_k(\mathbf{u}_k^* - \mathbf{v}_{k-}) = \mathbf{R}_k \quad (k = 1, \ldots, n)$$
$$\sum_k \mathbf{N}_{sk}\mathbf{u}_k^* + d_s = 0 \quad (s = 1, \ldots, l)$$

For the second phase we have the relations

$$m_k(\mathbf{v}_{k+} - \mathbf{u}_k^*) = \mathbf{R}_k^*$$

As carrier velocities we take any vectors that satisfy Eqs. (1.2) for links. The equations of impulsive motions are of the form

$$m_k(\mathbf{v}_{k+} - \mathbf{v}_{k-}) = \mathbf{R}_k + \mathbf{R}_k^*$$

The ideality of links under impact enables us to carry out further proof of the theorem in a manner similar to that presented above, since

$$\sum_k \mathbf{R}_k \mathring{\mathbf{v}}_{k-} = 0, \quad \sum_k \mathbf{R}_k^* \mathring{\mathbf{v}}_{k-} = 0$$

As the result, we have the following theorem. In the case of impulsive variation of ideal nonconservative links the acquired kinetic energy of the relative velocities is equal to the kinetic energy of acquired velocities.

We have, thus, obtained an extension of the Carnot theorem to systems with ideal acatastatic links under impact.

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